Colouring Cubic Graphs by Small Steiner Triple Systems

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Abstract. Given a Steiner triple system $\mathcal{S}$, we say that a cubic graph G is $\mathcal{S}$-colourable if its edges can be coloured by points of $\mathcal{S}$ in such way that the colours of any three edges meeting at a vertex form a triple of $\mathcal{S}$. We prove that there is Steiner triple system $\mathcal{U}$ of order 21 which is universal in the sense that every simple cubic graph is $\mathcal{U}$-colourable. This improves the result of Grannell et al. [J. Graph Theory **46** (2004), 15–24] who found a similar system of order 381. On the other hand, it is known that any universal Steiner triple system must have order at least 13, and it has been conjectured that this bound is sharp (Holroyd and Škoviera [J. Combin. Theory Ser. B **91** (2004), 57–66]).

Key words. cubic graph, edge-colouring, Steiner triple system

1. Introduction

The celebrated Vizing's edge-colouring theorem asserts that any cubic graph can be edge-coloured by three or four colours in such a way that adjacent edges receive distinct colours. While three colours are not enough to colour all cubic graphs, and the corresponding decision problem is difficult [9], edge-colourings by four or more colours are easy to deal with. It is therefore reasonable to focus on edge-colourings of cubic graphs where the occurrence of colours at vertices is controlled by a certain list $\mathcal{S}$ of permitted triples of colours. An edge-colouring according to such a list is then referred to as an *$\mathcal{S}$-colouring*. It is quite natural to take $\mathcal{S}$ to be a Steiner triple system because this choice offers a natural generalisation of the usual 3-edge-colouring which preserves the property that any two colours at a vertex determine the third colour. The study of such colourings was proposed by Archdeacon [1,2] in 1986 and is the main purpose of the present paper.

Recall that a *Steiner triple system* $\mathcal{S} = (X, T)$ of order n is a collection T of three-element subsets (called *triples* or *blocks*) of a set X of n *points* such that each pair of points is together present in exactly one triple. The smallest Steiner triple system is the *trivial* system $\mathcal{I}$ which has three points and a single triple. In general, a Steiner triple system of order n exists if and only if $n \equiv 1$ or 3 (mod 6), and such values of n are called *admissible*; for more information consult Colbourn and Rosa [4].

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In the context of edge-colourings, two infinite classes of Steiner triple systems are of particular interest. The *projective* Steiner triple system $PG(n,2)$, $n \geq 1$, has $\mathbb{Z}_2^{n+1} - \{0\}$ as its point set, the blocks of the system being the triples $\{x,y,z\}$ of points such that $x+y+z=0$. The *affine* Steiner triple system $AG(n,3)$, $n \geq 1$, has point set $\mathbb{Z}_3^n$, the triples of the system being again the triples with zero sum. The first of these classes includes the smallest non-trivial Steiner triple system $PG(2,2)$, the *Fano plane* of order 7. Observe that the trivial Steiner triple system $\mathcal{I}$ is included in both families as their first member.

It has been shown by Holroyd and Škoviera [8] that if $\mathcal{S}$ is the projective system $PG(n,2)$ with $n \geq 2$, then a cubic graph (with possibly parallel edges but no loops) is $\mathcal{S}$-colourable if and only if it is bridgeless, and that every bridgeless cubic graph has an $\mathcal{S}$-colouring for every non-trivial Steiner triple system.

Theorem 1. [8] *Let G be a cubic graph with no loops, and let $\mathcal{S} = PG(n,2)$ where $n \geq 2$. Then G is $\mathcal{S}$-colourable if and only if G is bridgeless.*

Theorem 2. [8] *Let G be a bridgeless cubic graph with no loops, and let $\mathcal{S}$ be a Steiner triple system of order greater than* 3. *Then G is $\mathcal{S}$-colourable.*

The study of Steiner colourings of cubic graphs with bridges is slightly more delicate. On the one hand, there are infinitely many Steiner triple systems which do not colour any cubic graph having a bridge (such as the projective systems), and, on the other hand, there are infinitely many cubic graphs with bridges which cannot be coloured by any Steiner triple system (for example, cubic graphs which contain a triangle with one doubled edge).

The reasons why some cubic graphs cannot be coloured by any Steiner triple system are easy to explain. If we are to colour a cubic graph G containing a pair of parallel edges, we can form a smaller cubic graph G' by contracting these two edges and suppressing the resulting 2-valent vertex. Observe that G is $\mathcal{S}$-colourable precisely when G' is. Thus if the repeated use of the above reduction eventually results in a cubic graph having a loop, we can conclude that G cannot be coloured by any Steiner triple system. The latter occurs precisely when G contains a bridge whose removal leaves a 2-connected component with no subdivision of the complete graph K_4. In this case we say that G has a *bridge with a series-parallel end*, because the corresponding 2-connected component is a series-parallel cubic graph with a single subdivided edge (see [6] for details).

Grannell et al. [7] have recently shown that the occurrence of a bridge with a series-parallel end in a cubic graph is the only obstruction to having an $\mathcal{S}$-colouring by some Steiner triple system $\mathcal{S}$. In fact, they proved that there is a single Steiner triple system $\mathcal{T}$ such that every cubic graph with no series-parallel end (in particular, every simple cubic graph) has a $\mathcal{T}$-colouring. To put it differently, $\mathcal{T}$ is a *universal* Steiner triple system, one which colours all Steiner colourable cubic graphs.

The system $\mathcal{T}$ has 381 points, but only a small part of them is used for colouring. Moreover, it is not highly symmetrical. Therefore it seems unlikely that $\mathcal{T}$ could be the smallest universal Steiner triple system. Indeed, our first main result improves the result of Grannell et al. [7] by displaying a point-transitive universal Steiner triple system of order 21.

Theorem 3. *Every cubic graph with no series-parallel end has a $\mathcal{U}$-colouring where $\mathcal{U}$ is the direct product $PG(2,2) \times \mathcal{I}$ of the Fano plane and the trivial Steiner triple system.*

As follows from the following discussion, our system is minimal in the sense that it does not contain any universal Steiner triple system of a smaller order.

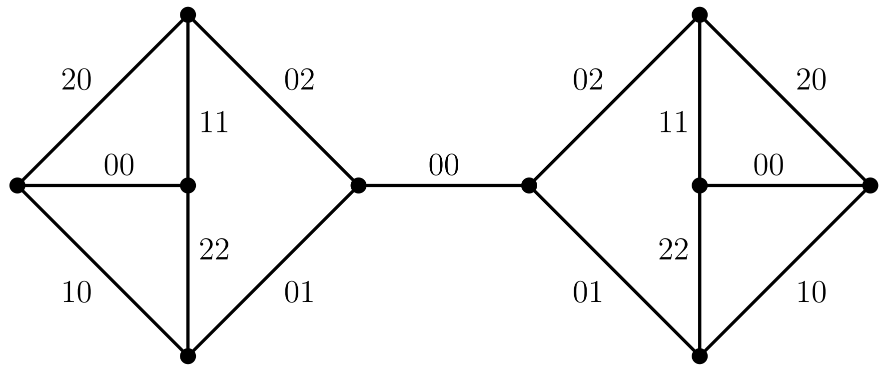

Fig. 1. A cubic graph with a bridge coloured by the affine Steiner triple system $AG(2, 3)$.

By Theorem 1, cubic graphs with bridges fail to have projective colourings. Therefore no projective Steiner triple system is universal. Although affine systems can colour certain cubic graphs with bridges (see Fig. 1), they do not colour all such graphs. For example, if a cubic graph contains a bridge whose removal yields a component which becomes bipartite after the suppression of the resulting 2-valent vertex, it cannot be coloured by any affine Steiner triple system [8, Theorem 1.3]. In such a case, the suppressed vertex is called a *bipartite end* of the bridge. Thus bridges with a bipartite end are an obstruction to affine colourings.

Since $PG(2, 2)$ and $AG(2, 3)$ are the only Steiner triple systems of order 7 and 9, respectively, all this implies that the order of any universal system is at least 13. On the other hand, a Steiner triple system of order n containing a Steiner triple system of order m must clearly have at least $2m + 1$ elements. Therefore all universal systems of order smaller than 27 are minimal. In addition, a result of Doyen and Wilson [5] implies that any Steiner triple system of order m may be embedded in a Steiner triple system of order n for any admissible $n \geq 2m + 1$. Consequently, Theorem 3 implies that for each admissible $n \geq 43$, there exists a universal Steiner triple system of order n.

In [8, Conjecture 1.4] Holroyd and Škoviera conjecture that bipartite ends of bridges are in fact the only obstruction to affine colourings, and moreover that all non-projective and non-affine Steiner triple systems are universal.

Conjecture 1. Let G be a cubic graph with no series-parallel end, and let $\mathcal{S}$ be a non-projective Steiner triple system. Then G fails to have an $\mathcal{S}$-colouring if and only if $\mathcal{S}$ is affine and G has a bridge with a bipartite end.

The conjecture claims, in particular, that every cubic graph without a bipartite end admits a colouring by any affine system $AG(n, 3)$ where $n \geq 2$. Although we are unable to prove this claim, we offer the following weaker result.

Theorem 4. *Every cubic graph with no bipartite end has an* $AG(n, 3)$*-colouring for each* $n \geq 3$.

We leave it as an open problem to prove that every cubic graph with no bipartite end has an $AG(2, 3)$-colouring.

2. Preliminaries

Given a Steiner triple system $\mathcal{R} = (P, B)$ of order m and a Steiner triple system $\mathcal{S} = (Q, C)$ of order n, we can construct a Steiner triple system of order mn, their *direct product* $\mathcal{R} \times \mathcal{S}$, as follows. We take the point-set of $\mathcal{R} \times \mathcal{S}$ to be $P \times Q$ and the triples to be all three-element sets of the form

- $\{(p, q_1), (p, q_2), (p, q_3)\}$, where $p \in P$ and $\{q_1, q_2, q_3\} \in C$,
- $\{(p_1, q), (p_2, q), (p_3, q)\}$, where $\{p_1, p_2, p_3\} \in B$ and $q \in Q$, and
- $\{(p_1, q_1), (p_2, q_2), (p_3, q_3)\}$, where $\{p_1, p_2, p_3\} \in B$ and $\{q_1, q_2, q_3\} \in C$.

This definition extends to the direct product of any finite number of Steiner triple systems. For example, the direct product of n copies of the trivial system $\mathcal{I}$ is isomorphic to the affine Steiner triple system $AG(n, 3)$. It is therefore convenient to treat the trivial Steiner triple system as the affine system $AG(1, 3)$ with point-set $\mathbb{Z}_3$.

Given a Steiner triple system $\mathcal{S}$, an $\mathcal{S}$-colouring of a cubic graph G is a colouring of the edges of G by points of $\mathcal{S}$ such that the colours of any three edges meeting at a vertex form a triple of $\mathcal{S}$. (In case $\mathcal{S}$ is the Fano plane, we simply speak of a *Fano* colouring.) A *weak* $\mathcal{S}$-colouring allows the colours at a vertex either to be identical or to form a triple of $\mathcal{S}$. We call a vertex *regular* (or *strong*) if the colours of the incident edges form a triple of $\mathcal{S}$; otherwise we call it *weak*. Observe that if ϕ is a weak $\mathcal{R}$-colouring and ψ is a weak $\mathcal{S}$-colouring of a cubic graph G, then the product mapping $(\phi, \psi) : e \mapsto (\phi(e), \psi(e))$ is a weak $\mathcal{R} \times \mathcal{S}$-colouring. A vertex v of G is weak in (ϕ, ψ) if and only if it is weak in both ϕ and ψ. Thus the product of two weak colourings with disjoint sets of weak vertices is a regular colouring.

The concept of an $\mathcal{S}$-colouring naturally extends to graphs with maximum valency three. In a regular colouring we simply require the colours meeting at a vertex to be distinct elements of a block of $\mathcal{S}$, and in a weak colouring we allow all colours at a vertex to be equal.

Steiner colourings are sometimes related to flows on graphs. To define a flow on a graph G, let $D(G)$ denote the set which is obtained by replacing each edge of G with a pair of oppositely directed *darts*; we call $D(G)$ the *dart-set* of G. Each dart z, including those on loops, has its inverse dart $z^{-1} \neq z$ which is incident with the same vertices but has opposite direction. For an arbitrary vertex v, we let $D(v)$ be the set of all darts emanating from v. Clearly, these sets partition the whole dart-set.

Let A by any Abelian group, written additively. We define an *A-chain* on G to be a function $\xi : D(G) \to A$ satisfying the following condition:

(F1) $\xi(z^{-1}) = -\xi(z), \quad$ for each dart $z \in D(G)$.

For a vertex v, let $\partial\xi(v) = \sum_{z \in D(v)} \xi(z)$. This value is the *outflow* from v. A vertex with a non-zero outflow will be called *singular*. An A-chain ξ is an *A-flow* if it has no singular vertices, that is, if the following “flow-conservation property” holds:

(F2) $\partial\xi(v) = 0, \quad$ for each vertex $v \in V(G)$.

A flow is said to be *nowhere-zero* if $\xi(z) \neq 0$ for each dart $z \in D(G)$. (For example, any $\mathcal{S}$-colouring where $\mathcal{S} = PG(n, 2)$, $n \geq 2$, is a nowhere-zero $\mathbb{Z}_2^{n+1}$-flow.)

Observe that if every element of A is self-inverse, then $\xi(z) = \xi(z^{-1})$ for each dart z, and we may simply view an A-flow on G as a function defined on the edges of G rather than on darts. Note that in this case the group A will be isomorphic to a direct product of copies of $\mathbb{Z}_2$.

3. Proofs

In this section we prove our main results, Theorem 3 and Theorem 4. We start with the observation that it is sufficient to prove these results for simple graphs. Indeed, any cubic graph G

with no series-parallel end can be reduced to a simple cubic graph which is $\mathcal{S}$-colourable by a Steiner triple system $\mathcal{S}$ if and only if G is. A triple edge can be simply removed from the graph, and a pair of parallel edges can be contracted to a vertex which can be subsequently suppressed. If these two operations are repeated sufficiently many times, the result will be a simple cubic graph G' with the required property. *In the rest of this section we therefore assume G to be simple.*

The main idea of both proofs is to decompose G into a set of blocks, and for each block to construct a pair of weak Steiner colourings such that their sets of weak vertices are disjoint. For the first coordinate we use the Fano plane in the proof of Theorem 3, and the affine plane $AG(2,3)$ in the proof of Theorem 4. For the second coordinate we use the trivial system in the proofs of both theorems. Finally, we merge all the coloured blocks together.

Throughout this section, by a *block* of a cubic graph G we mean a connected component of the graph obtained from G by removing all its bridges. This definition slightly deviates from the standard notion of a block but is better suited for our aims.

Let B be a block of a cubic graph G. Denote by B^+ the subgraph obtained from B by attaching all incident bridges, and by B^- the graph obtained from B by suppressing the 2-valent vertices of B. Thus B^- is a bridgeless cubic graph, although not necessarily simple.

We distinguish between three types of blocks. A *trivial* block consists of a single vertex which is incident with three bridges. Non-trivial blocks split into those which contain a bipartite end and those which have no bipartite end. The latter blocks will be called *standard*. Standard blocks can have any number of incident bridges. However, a block containing a bipartite end has exactly one incident bridge because no bipartite cubic graph can have a bridge.

Trivial blocks can be managed easily: we simply colour the three incident bridges by any triple of the Fano plane and by the only triple of the trivial Steiner triple system. Non-trivial blocks are handled in the following four lemmas.

Lemma 1. *Let B be a non-trivial block of a simple cubic graph, and let $\mathcal{S}$ be any non-trivial Steiner triple system. Then B^+ has a weak $\mathcal{S}$-colouring such that each weak vertex is a bridge end.*

Proof. Consider the graph B^-. Since B^- is bridgeless, Theorem 2 implies that it admits an $\mathcal{S}$-colouring. We reinsert the bridges and assign each bridge and the other two adjacent edges of B^+ the colour of the corresponding edge of B^-. The result is a weak $\mathcal{S}$-colouring of B^+ whose weak vertices are exactly the bridge ends in B.

Since a bipartite cubic graph is 3-edge-colourable, the same argument yields a weak colouring by the trivial Steiner triple system for each block B such that B^- is bipartite.

Lemma 2. *Let B be a non-trivial block of a simple cubic graph such that B^- is bipartite. Then B^+ has a weak $\mathcal{I}$-colouring such that each weak vertex is a bridge end.*

Lemma 3. *Let B be a standard block of a simple cubic graph. Then B^+ has a weak $\mathcal{I}$-colouring such that all bridge ends in B are regular vertices.*

Proof. If B^+ has no bridges, then the all-zero $\mathcal{I}$-colouring suffices. If B^+ does have some bridges, we consider two cases.

Case 1. Let B^+ have exactly one bridge, and let v be its end in B. To find the required colouring we first show that B contains an even cycle through v. Since v is not a bipartite end, B contains either an odd cycle avoiding v or an even cycle containing v. In the former case, take such an odd cycle C and choose two internally disjoint paths P and Q of minimum length which connect v to C. Their respective end-vertices on C divide C into two paths C' and C''. As the lengths of these paths are of different parity, one of the cycles $PC'Q$ and $PC''Q$ is even and passes through v. Thus B has an even cycle through v in each case.

The required colouring is now easy to construct. We colour the edges of an even cycle through v alternately 1 and 2 and let all other edges of B^+ carry the colour 0. The result is a weak $\mathcal{I}$-colouring of B^+ under which the vertex v will be regular.

Case 2. Assume that B^+ has more than one bridge. Denote the bridges by u_1v_1, $u_2v_2, \ldots,$ u_mv_m in such a way that each v_i lies in B and the other end-vertex u_i lies outside B. For each u_i take the nearest u_j, $j \neq i$, and let P_i be the shortest path from u_i to u_j. Clearly, no u_l or v_l can lie on P_i if $l \neq i$ and $l \neq j$. We now construct a sequence of weak colourings $\phi_0, \phi_1, \ldots, \phi_m$ of B^+ starting with the all-zero colouring ϕ_0. The colouring ϕ_m will be the colouring sought.

Assume that we have already constructed a colouring ϕ_{i-1} under which all the vertices v_k with $1 \leq k \leq i-1$ are regular. If v_i is a regular vertex with respect to ϕ_{i-1}, we set $\phi_i = \phi_{i-1}$. Otherwise, if v_i is a weak vertex, we recolour B^+ along the path P_i as follows. We construct an auxiliary weak colouring ψ_i that is all-zero except on P_i where we assign the colours 1 and 2 alternately, either starting with 1 or starting with 2. The initial colour on P_i will be chosen in such way that it will make both v_i and the other vertex v_j on P_i regular with respect to ϕ_i. This is always possible: if v_j is weak, both possibilities for ψ_i are feasible, but when v_j is regular, only of them will be good and the other possibility will make v_j a weak vertex. By setting $\phi_i(e) = \phi_{i-1}(e) + \psi_i(e)$ for each edge e of B^+ we obtain a correct weak colouring, because at any internal vertex w of P_i the sum of colours on the three edges x, y and z incident with w remains zero. If, for example, x and y lie on P_i, then indeed

$$\begin{aligned}\phi_i(x) + \phi_i(y) + \phi_i(z) &= \phi_{i-1}(x) + \psi_i(x) + \phi_{i-1}(y) + \psi_i(y) + \phi_{i-1}(z) \\ &= \phi_{i-1}(x) + \phi_{i-1}(y) + 1 + 2 + \phi_{i-1}(z) = 0.\end{aligned}$$

This completes the proof.

Recall that a Steiner triple system is said to be *point-transitive* if for any two points x and y there exists an automorphism mapping x to y. If for any two ordered pairs of distinct points (x, y) and (z, w) there is an automorphism taking x to y and z to w, the system is said to be 2-*point-transitive*. The trivial system is obviously 2-point-transitive and so are all affine and projective Steiner triple systems. Their automorphisms are respectively the affine transformations and the collineations of the corresponding geometric spaces (see, for example, Biggs and White [3, Chapter 2]). It is easy to see that the direct product of two 2-point-transitive Steiner triple systems is 2-point-transitive too.

We proceed to our last lemma.

Lemma 4. *Let G be a simple cubic graph and let B be a block of G containing a bipartite end. Then B^+ has a weak Fano colouring such that the bipartite end is regular.*

Proof. We employ induction on the number of vertices of B^+. The graph B^+ with the least number of vertices is the complete bipartite graph $K_{3,3}$ with a bridge attached to one of its edges. A weak Fano colouring of this graph is displayed in Fig. 2. This forms the basis of the induction.

Now let B be a block with a bipartite end v such that B^- has order greater than 6. Let x and y be the neighbours of v in B, and let h be the edge connecting x to y in B^-. Clearly, the vertex-set of B^- has a bipartition $X \cup Y$ such that $x \in X$ and $y \in Y$. Its edge-set can thus be

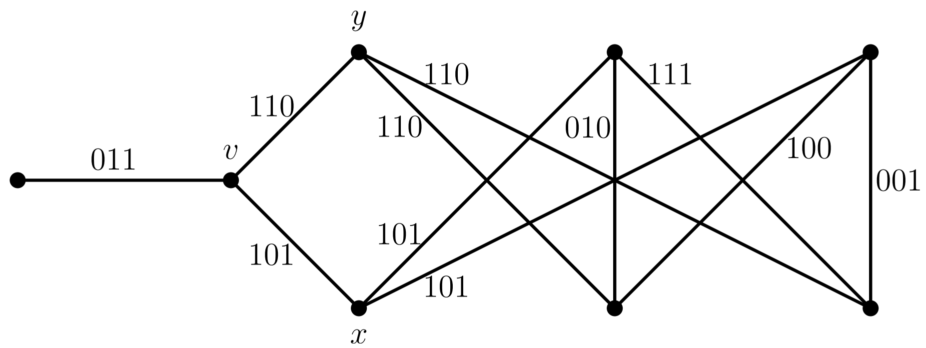

Fig. 2. A weak Fano colouring of $K_{3,3}$ with a subdivided edge and a bridge.

partitioned into three disjoint 1-factors F_1, F_2 and F_3 where F_1 denotes the one that contains the edge h. Let us colour the edges of F_1, F_2 and F_3 respectively by the colours $(0,1,1)$, $(0,0,1)$ and $(0,1,0)$. This colouring forms a nowhere-zero $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$-flow on B^-. We first transform this flow into a $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$-chain ϕ on B^+ as follows. We colour the bridge at v by $(0,1,1)$, the edge vx by $(1,0,1)$, and the edge vy by $(1,1,0)$. We leave all the colours of the remaining edges untouched. Note that v and all other vertices except x and y fulfil the flow-conservation property (F2) and hence are regular. The vertices x and y, however, are neither regular nor weak: they are singular with outflow $(1,1,0)$ and $(1,0,1)$, respectively. Our aim is to correct this anomaly by sending certain flow values from x and y along suitable paths while keeping the regularity of v and non-zero values on all edges of B^+. By sending a value $g\in\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ along a path P we mean that we form a new $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$-chain ϕ' on B^+ such that $\phi'(e)=\phi(e)+g$ for each edge e of P, and $\phi'(e)=\phi(e)$ otherwise. In doing this we may move a singularity to another vertex or create a new singular vertex, but eventually these singular vertices will be weak and the result will be a weak Fano colouring of B^+.

First of all observe that $F_2\cup F_3$ forms an even 2-factor of B^-, that is, a set of disjoint even cycles covering all the vertices. Direct all the edges of F_1 from X to Y and contract the cycles of the 2-factor to obtain a new graph H. This graph is endowed with an orientation under which each vertex has the same number of incoming and outgoing directed edges. Thus H admits an Euler trail following the orientation of H and therefore contains no edge-cuts of odd size. Note, however, that H may have both parallel edges and loops.

For any vertex p of B^-, let $\bar{p}$ denote the corresponding vertex of H. It is easy to see that every $\bar{p}$-$\bar{q}$-path P in H gives rise to a p-q-path $\tilde{P}$ in B^- which alternately uses the edges of P and segments of cycles of the complementary 2-factor $F_2\cup F_3$.

To establish the existence of the required weak Fano colouring we consider a number of cases.

Case 1. Let H have a single vertex. Then B^- is a hamiltonian graph with $F_2\cup F_3$ forming a Hamilton cycle. Divide $F_2\cup F_3$ into a union of an x-y-path P and a y-x-path Q; without loss of generality we may assume that P begins (and ends) with an F_2-edge and that Q begins (and ends) with an F_3-edge. Consider the set of all edges in $F_1-\{h\}$ which have an end on P. There are three types of such edges:

- edges with both ends on P (called *Type* 1);
- edges with one end in $P\cap X$ and the other on Q (called *Type* 2); and
- edges with one end in $P\cap Y$ and the other on Q (called *Type* 3).

Because B^- is bipartite, there is an equal number of edges of Type 2 and Type 3.

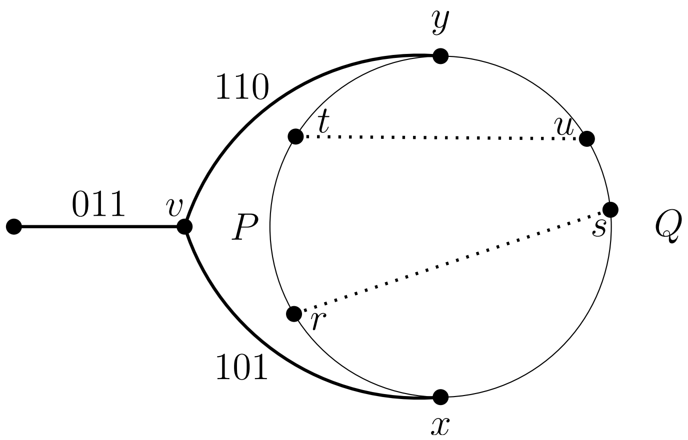

Fig. 3. A Hamilton cycle of B^- embedded in B^+. Two important edges rs and tu shown dotted may interlace.

Subcase 1.1. Assume that there exists an edge rs of Type 2 and an edge tu of Type 3. Let r and t be their end-vertices on P. Then $C_1 = P(x,r)rsQ(s,x)$ and $C_2 = P(x,t)tuQ(u,x)$ are two distinct cycles through x but not through y (see Fig. 3). We send the value $(1,1,1)$ along P and the value $(1,0,0)$ along Q. This makes the vertex y weak. Further, we send the value $(0,0,1)$ along C_1 and the value $(0,1,0)$ along C_2. As a result, the vertex x becomes weak, too. It is easy to see that all the vertices except x and y remain regular and that no edge receives the value $(0,0,0)$. Thus we have obtained a valid weak Fano colouring as claimed. (The just described construction works also for the induction basis, and the colouring obtained here is exactly the same as that described in Fig. 2.)

Subcase 1.2. There are no edges of Type 2 and Type 3. Since B^+ is simple, this can only happen when each of P and Q has length at least 7. Let x' be the successor of x on P, and let y' be the predecessor of y on P. Send the value $(1,1,0) = \partial\phi(x)$ along the edge xx' and the value $(1,0,1) = \partial\phi(y)$ along the edge yy'. The edges xx' and yy' will now be coloured $(1,1,1)$ and $(1,0,0)$, and the vertices x and y become regular. Clearly, $\{xx', yy'\}$ is an edge-cut in B^+ which separates x and y from x' and y'. In the latter component, let us connect x' and y' to a new vertex v' and add a pendant edge at v'. We have thus formed a smaller block D with a bipartite end v'. Moreover, D is a simple graph. By the induction hypothesis, D^+ admits a weak Fano colouring such that v' is regular. Since the Fano plane is 2-transitive, we can arrange the colouring in such a way that $v'x'$ is coloured $(1,1,1)$ and $v'y'$ is coloured $(1,0,0)$. By removing v' from D and pasting the rest of D to its original place in B^+ we obtain a weak Fano-colouring of the whole B^+.

Case 2. Let H have at least two vertices and $\bar{x} \neq \bar{y}$. We distinguish two subcases.

Subcase 2.1 Assume that the vertices $\bar{x}$ and $\bar{y}$ are separated by a 2-edge-cut. Since the edge h joins $\bar{x}$ to $\bar{y}$ in H, one of these edges is h. Let k be the other one, and let u and w be its end-vertices in B^+. Remove h and k from H and let M be the component containing $\bar{x}$ and one of $\bar{u}$ and $\bar{w}$, say $\bar{u}$; let N be the component containing $\bar{y}$ and $\bar{w}$. Since h and k lie on a directed Euler trail, k is directed from N to M forcing u to be a member of the partite set Y. Take a $\bar{y}$-$\bar{u}$-path in H avoiding h. The corresponding y-u-path P in B^+ necessarily uses k as its last edge. Send the value $(1,0,1) = \partial\phi(y)$ along P from y to u. This will change the colour of k to $(1,1,0)$, making y a regular vertex and u a singular vertex with outflow $(1,0,1)$. We now remove the edges xv and k from B^+ and connect x and u to a new vertex v' to form a smaller block with a bipartite end. By using the induction hypothesis and a similar cut-and-paste construction as in Subcase 1.2 we obtain the desired weak colouring of B^+.

Subcase 2.2 There is no 2-edge-cut separating $\bar{x}$ from $\bar{y}$. Since H has a directed Euler trail and the edge h is directed from $\bar{x}$ to $\bar{y}$, it is easy to see that there must be a directed $\bar{x}$-$\bar{y}$-path R in H which avoids the edge h. In B^+, the last edge of R terminates at a vertex $w \neq y$ which lies on the same cycle C of $F_2 \cup F_3$ as y does. Clearly, $w \in Y$. Divide C into two w-y-paths P and Q in such a way that P starts with an F_2-edge and Q starts with an F_3-edge. Now let us send the value $(1,1,0) = \partial\phi(x)$ along any x-w-path $\tilde{R}$ in B^+ which corresponds to R. This turns x into a regular vertex and w into a singular vertex at which the incident F_1-edge is coloured $(1,0,1)$. To make w a weak vertex, let us send the value $(1,0,0)$ along the path P and the value $(1,1,1)$ along the path Q. As a result, the vertex y becomes weak too, and the new $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-chain is a weak colouring of B^+.

Case 3. Let H have at least two vertices and let $\bar{x} = \bar{y}$. Again there are two subcases to consider.

Subcase 3.1. First assume that H contains a 2-edge-cut, say $S = \{k, l\}$. Let u and w be the end-vertices of the edges k and l in B^+ such that the corresponding vertices $\bar{u}$ and $\bar{w}$ belong to the component of $H - S$ not containing the vertex $\bar{x}$. The orientation of H indicates that u and w lie in different partite sets of B^-. Since H is 2-edge-connected, there exist paths P and Q in B^- from x to u and from y to w, respectively, whose common edges are contained in $F_2 \cup F_3$. We now send the value $(1,1,0) = \partial\phi(x)$ along P and the value $(1,0,1) = \partial\phi(y)$ along Q. By the choice of P and Q, no edge will receive the value $(0,0,0)$, and k will be coloured $(1,0,1)$ while l will be coloured $(1,1,0)$. At this point we can remove k and l from B^+ and continue as in Subcase 1.2.

Subcase 3.2. The graph H is 4-edge-connected. Pick any vertex p of H other than $\bar{x}$. Then $\bar{x}$ is connected to p by four edge-disjoint paths. From the pigeonhole principle, at least two of the paths must have their last edges oriented consistently, that is, either both from p or both to p. It follows that in B^+ there exist two paths P and Q from x and y, respectively, to certain vertices u and w which lie in a common cycle C of $F_2 \cup F_3$ and belong to the same partite set of B^-. Again, the common edges of P and Q are contained in $F_2 \cup F_3$. We now send the value $(1,1,0) = \partial\phi(x)$ along the path P and the value $(1,0,1) = \partial\phi(y)$ along Q to make the vertices x and y regular and the vertices u and w singular. By the choice of P and Q, no edge has received the value $(0,0,0)$. The F_1-edges at u and v are now coloured $(1,0,1)$ and $(1,1,0)$, respectively. To make these vertices weak, we divide C into two u-w-paths and send the value $(1,0,0)$ along the path which starts with an F_2-edge and the value $(1,1,1)$ along the path which starts with an F_3-edge. The result is clearly a weak Fano colouring of B^+.

In all the above cases we have constructed a weak Fano colouring of B^+, hence the lemma is proved.

Proof of Theorem 3. As noted earlier in this section, we can assume G to be simple. We can also assume that G is connected.

Let $B_1, B_2, \ldots, B_k$ be the blocks of G ordered in such way that for each $i \geq 1$ there is exactly one bridge connecting B_{i+1} to $B_1 \cup B_2 \cup \cdots \cup B_i$. Such an ordering always exists because contracting each block into a vertex results in a tree.

We now colour the graphs B_i^+ step by step with the increasing index. We do so by finding a weak Fano colouring and a weak $\mathcal{I}$-colouring which combine into a proper $\mathcal{U}$-colouring where $\mathcal{U} = PG(2,2) \times \mathcal{I}$.

– If B_i is a standard block, we construct a weak Fano colouring ϕ of B_i^+ whose weak vertices are bridge ends in B_i, and a weak $\mathcal{I}$-colouring ψ under which the bridge ends are regular. These colourings are guaranteed by Lemma 1 and Lemma 3, respectively. The pair (ϕ, ψ) is clearly a proper $\mathcal{U}$-colouring of B_i^+.
– If the block B_i contains a bipartite end, we construct a weak Fano colouring ϕ of B_i^+ such that the bipartite end is regular and a weak $\mathcal{I}$-colouring ψ under which the bipartite end is weak but all other vertices are regular. Such colourings exist by Lemma 4 and Lemma 2, respectively. Again, (ϕ, ψ) is a proper $\mathcal{U}$-colouring of B_i^+.
– If B_i is a trivial block, we simply colour B_i^+ by a suitable triple from $\mathcal{U}$.

Now we merge the colourings together. Assume that the graph $B_1^+ \cup \cdots \cup B_j^+$, $j \geq 1$, has already been coloured as indicated above. By the ordering of the blocks, exactly one edge of B_{j+1}, necessarily a bridge, has received a colour in one of the previous steps. Since the system $\mathcal{U}$ is point-transitive, we can transform any $\mathcal{U}$-colouring of B_{j+1}^+ into one where the specified bridge is coloured by the required colour. This colouring can then be combined with the colouring of $B_1^+ \cup \cdots \cup B_j^+$ into a $\mathcal{U}$-colouring of $B_1^+ \cup \cdots \cup B_{j+1}^+$. By continuing this process as long as necessary we eventually colour the whole of G. □

Proof of Theorem 4. The system $AG(3,3)$ isomorphically embeds into each affine system $AG(n,3)$ with $n \geq 3$, so it is sufficient to construct an $AG(3,3)$-colouring for each cubic graph G with no bipartite end. As in the proof of the previous theorem we assume G to be simple and connected.

Since $AG(3,3)$ is point-transitive and can be expressed as $AG(2,3) \times \mathcal{I}$, we can proceed analogously as in the previous proof except the following. Let B_i be a non-trivial block of G. As B_i does not have a bipartite end, the graph B_i^+ admits a weak $AG(2,3)$-colouring ϕ whose weak vertices are bridge ends in B_i, and a weak $\mathcal{I}$-colouring ψ under which all bridge ends in B_i are regular vertices. These colourings exist by Lemma 1 and Lemma 3, respectively. Clearly, (ϕ, ψ) is a proper $AG(3,3)$-colouring of B_i^+.

The proof can now be finished in exactly the same manner as the previous one. □

Remark. It may be interesting to note that the universal Steiner triple system $\mathcal{T}$ constructed by Grannell et al. in [7] contains a copy of our system $\mathcal{U} = PG(2,2) \times \mathcal{I}$. The fact that $\mathcal{T}$ is indeed universal thus follows from our Theorem 3.

References

1. D. S. Archdeacon, *Coverings of graphs by cycles*, Congr. Numer. **53** (1986), 7–14.

2. D. S. Archdeacon, *Generalizations of Tait colouring cubic graphs*, Problems in Topological Graph Theory, `http://www.emba.uvm.edu/~archdeac/ problems/petecol.htm`

3. N. L. Biggs and A. T. White, *Permutation Groups and Combinatorial Structures*, Cambridge Univ. Press, Cambridge, 1979.

4. C. J. Colbourn and A. Rosa, *Triple Systems*, Oxford Univ. Press, Oxford, 1999.

5. J. Doyen and R. M. Wilson, *Embeddings of Steiner triple systems*, Discrete Math. **5** (1973), 229–239.

6. R. J. Duffin, *Topology of series-parallel networks*, J. Math. Analysis and Appl. **10** (1965), 303–318.

7. M. J. Grannell, T. S. Griggs, M. Knor, and M. Škoviera, *A Steiner triple system which colors all cubic graphs*, J. Graph Theory **46** (2004), 15–24.

8. F. C. Holroyd and M. Škoviera, *Colouring of cubic graphs by Steiner triple systems*, J. Combin. Theory Ser. B **91** (2004), 57–66.

9. I. Holyer, *NP-completness of edge-colouring*, SIAM J. Comp. **10** (1981), 718–720.

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